

Self-similar Approximants of the Permeability in Heterogeneous Porous Media from Moment Equation Expansions

S. Gluzman¹ and D. Sornette^{1,2,3}

¹ Institute of Geophysics and Planetary Physics
University of California Los Angeles, Los Angeles, CA 90095-1567

² Department of Earth and Space Sciences, UCLA

³ Laboratoire de Physique de la Matière Condensée
CNRS UMR 6622 and Université de Nice-Sophia Antipolis, 06108 Nice Cedex 2, France

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Abstract

We use a mathematical technique, the self-similar functional renormalization, to construct formulas for the average conductivity that apply for large heterogeneity, based on perturbative expansions in powers of a small parameter, usually the log-variance σ_Y^2 of the local conductivity. Using perturbation expansions up to third order and fourth order in σ_Y^2 obtained from the moment equation approach, we construct the general functional dependence of the transport variables in the regime where σ_Y^2 is of order 1 and larger than 1. Comparison with available numerical simulations give encouraging results and show that the proposed method provides significant improvements over available expansions.

1 Introduction

Subsurface hydraulic parameters such as medium permeability and porosity, saturation curves and relative permeability have been traditionally viewed as well-defined local quantities that can be assigned unique values at each point in space. Yet, subsurface flow takes place in a complex environment whose makeup varies in a manner that cannot be predicted deterministically in all of its relevant details. This makeup tends to exhibit discrete and continuous variations on a multiplicity of scales, causing hydraulic parameters to do likewise. In practice, such parameters can at best be measured at selected locations and depth intervals where their values depend on the scale (support volume) and mode (instrumentation and procedure) of measurement. Estimating the parameters at points where measurements are not available entails a random error. Quite often, the support of measurement is uncertain and the data are corrupted by experimental and interpretive errors.

Uncertainty is usually dealt with either deterministically through upscaling or stochastically through the evaluating statistical moments. Statistical moments can be obtained through Monte Carlo simulations or development of moment differential equations, which is the method from which we start our analysis.

In the stochastic approach, parameter values determined at various points within a more-or-less distinct soil unit can be viewed as a sample from a random field defined over a continuum. This random field is characterized by a joint (multivariate) probability density function or, equivalently, its joint ensemble moments. Thus, a parameter such as (saturated, natural) log hydraulic conductivity $Y(\mathbf{x}) = \ln K_s(\mathbf{x})$ varies not only across the real space coordinates \mathbf{x} within the unit, but also in probability space (this variation may be represented by another “coordinate” ξ , the configuration coordinate, which, for simplicity, we suppress). Whereas spatial moments are obtained by sampling $Y(\mathbf{x})$ in real space (across \mathbf{x}), ensemble moments are defined in terms of samples collected in probability space (across ξ).

In the moment equation approach, which we propose to exploit here, the stochastic differential equations are averaged first to obtain moments differential equations (MDEs) governing the statistical moments of the dependent variables. The MDEs are themselves deterministic and can be solved numerically or sometimes, analytically. The MDE approach has important advantages. First, only a small number of equations must be solved: one for the mean and one each for a small number of variances and covariances. Second, the coefficients of the MDEs are relatively smooth because they are averaged quantities. Thus the MDEs can be solved on comparatively smooth grids. Third, the MDEs are available in analytical form, even though they are usually solved numerically in applications. This holds the potential for increased physical understanding of the mechanisms of uncertainty through qualitative analysis. Finally, in many applications MDE approaches provide a good estimate of the behavior of large variance systems despite being based on small perturbation theory.

Since the moment equation approach derives the equations of evolutions of the moments of the distribution of the transport variables by averaging the stochastic differential equations of transport in heterogeneous porous media, its fundamental limitation is the assumption that the variance σ_Y^2 of the log hydraulic conductivity is small, in contradiction with real geological settings of interest to a large variety of geophysical applications. In order to obtain reliable descriptions of the transport coefficients for large σ_Y^2 , our goal in the present paper is to adapt and extend the self-similar functional renormalization method to the moment equation approach. In essence fundamentally non-perturbative, this recently developed technique provides us with a stable and robust estimation of the transport variables at large values of the perturbation parameter σ_Y^2 . The functional renormalization method associates ideas from the renormalization group theory of multiscale and critical phenomena [21] with methods from the theory of dynamical systems and of control theory. Using perturbation expansions up to third order and fourth order in σ_Y^2 obtained from the moment equation approach, we construct the general functional dependence of the transport variables in the regime where σ_Y^2 is of order 1 and larger than 1.

The next section 2 recalls briefly how perturbation expansions are obtained from the moment equation approach. Section 3 summarizes the general formulation of the self-similar approximation theory. Section 4 gives the results of the application of the functional renormalization method to the moment equation expansions at increasing orders in σ_Y^2 . Section 5 formulates the expansion in powers of $1/d$, where d is the space dimension. This procedure well-known

in statistical physics is resummed by the functional renormalization to provide accurate formulas. Section 6 briefly outlines future directions of investigations.

2 Problem Formulation and perturbation expansions

2.1 Basic equations

It has become common to quantify uncertainty in ground water flow models by treating hydraulic conductivity, K , and derived quantities like hydraulic head, h , as random fields. For steady-state flows in the absence of sources and sinks, the statistics of h can be obtained from the stochastic flow equation

$$\nabla \cdot [K(\mathbf{x}) \nabla h(\mathbf{x})] = 0 \quad (1)$$

when the statistics of K are known. We further assume that the site of interest is sufficiently characterized so that available experimental data are sufficient to obtain the statistics of K , such as its (ensemble) mean, \overline{K} , variance, σ_K^2 , and (two-point) correlation structure, $\rho_K(\mathbf{x}, \mathbf{y})$. Then one can solve directly for the moments of h by developing deterministic equations for the moments from (1). In general this involves taking the expected value of (1) and similar equations for higher-order moments, closing the system of moment equations (usually through perturbation approximations). Numerical solutions for moment equations are typically computationally more efficient than Monte Carlo simulations. In the first place, taking expected values smoothes parameters in the moment equations which in turn allows low-resolution grids for numerical solutions. Furthermore, the number of moment equations is much smaller than the number of realizations required by Monte Carlo simulations. Additionally, the moment equations lend themselves to qualitative analysis.

Here, we concentrate on flow through highly heterogeneous porous media with the variance σ_Y^2 of log hydraulic conductivity $Y = \ln K$ as the expansion parameter of the theory. We estimate the mean hydraulic head, $\overline{h}(\mathbf{x})$, and assess the errors associated with such an estimation. We represent $K(\mathbf{x}) = \overline{K}(\mathbf{x}) + K'(\mathbf{x})$ as the sum of a mean, $\overline{K}(\mathbf{x})$, and a zero-mean random deviation, $K'(\mathbf{x})$, with variance $\sigma_K^2(\mathbf{x})$. Similarly, $h(\mathbf{x}) = \overline{h}(\mathbf{x}) + h'(\mathbf{x})$ with $\overline{h}'(\mathbf{x}) \equiv 0$ and variance $\sigma_h^2(\mathbf{x})$.

The average steady-state flow equation becomes

$$\nabla \cdot [\overline{K}(\mathbf{x}) \nabla \overline{h}(\mathbf{x})] + \nabla \cdot \overline{\mathbf{r}}(\mathbf{x}) = 0 \quad (2)$$

which consists of a deterministic mean part, $\overline{K} \nabla \overline{h}$, and a deterministic residual flux, $\overline{\mathbf{r}} = -\overline{K' \nabla h'}$. Solutions of (2) require the mean conductivity, $\overline{K}(\mathbf{x})$, and in most cases, a method for closing an expansion of $\overline{\mathbf{r}}(\mathbf{x})$. Usually $\overline{\mathbf{r}}(\mathbf{x})$ is approximated through perturbation expansions based on σ_Y^2 , the variance of $Y = \ln K$, the logarithm of conductivity. This approach works well as long as σ_Y^2 is small. This restriction is a stumbling block on the road to applicability of numerous theoretical analyses to real-world problems. Our goal here is to provide a general theoretical method to extend the domain of application of moment equations to the large heterogeneity limit.

2.2 Perturbation Expansions

Consider asymptotic expansions of the parameters and functions, $\overline{K} = K_g (1 + \sigma_Y^2/2 + \dots)$; $\overline{\mathbf{q}} = \overline{\mathbf{q}}^{(0)} + \overline{\mathbf{q}}^{(1)} + \dots$; $\overline{h} = \overline{h}^{(0)} + \overline{h}^{(1)} + \dots$; and $\mathbf{r} = \mathbf{r}^{(1)} + \dots$, where $K_g = \exp(\overline{Y})$, \overline{Y} being the ensemble mean of Y . The superscript (i) denotes terms that are of i th-order, i.e. contain only the i th power of σ_Y^2 . The first-order (in σ_Y^2) approximation of the residual flux is given by [22, and references therein]

$$\mathbf{r}^{(1)}(\mathbf{x}) = K_g \sigma_Y^2 \int_{\Omega} \rho_Y(\mathbf{y}, \mathbf{x}) \nabla_{\mathbf{x}} \nabla_{\mathbf{y}}^T G(\mathbf{y}, \mathbf{x}) \nabla \overline{h}^{(0)}(\mathbf{y}) d\mathbf{y}, \quad (3)$$

where $\rho_Y(\mathbf{y}, \mathbf{x})$ is the spatial two-point autocorrelation function of Y , and $G(\mathbf{y}, \mathbf{x})$ is the deterministic Green's function for Laplace equation in Ω subject to the corresponding homogeneous boundary conditions. It is a standard

practice in stochastic hydrogeology to rely on the first-order approximation of \mathbf{r} [2], but higher-order approximations are also available [13].

Collecting the terms of the same powers of σ_Y^2 yields the zeroth-order approximation of the mean head in 2,

$$K_g \nabla^2 \bar{h}^{(0)}(\mathbf{x}) = 0, \quad (4)$$

and its first-order approximation,

$$K_g \nabla^2 \bar{h}^{(1)}(\mathbf{x}) + \nabla \cdot \left[\frac{\sigma_Y^2}{2} K_g \nabla \bar{h}^{(0)}(\mathbf{x}) - \mathbf{r}^{(1)}(\mathbf{x}) \right] = 0. \quad (5)$$

Solving a system of these sequential approximations leads to $\bar{h}^{[1]} \equiv \bar{h}^{(0)} + \bar{h}^{(1)}$. Strictly speaking, for such expansions to be asymptotic it is necessary that $\sigma_Y^2 \ll 1$, i.e. that porous media be mildly heterogeneous. However, various numerical simulations (e.g., [12]) have demonstrated that these first-order approximations remain remarkably robust even for strongly heterogeneous media with σ_Y^2 as large as 4.

2.3 Effective Conductivity of Porous Media

For an effective conductivity to exist in the strict sense, it is necessary that $\nabla \bar{h}$ be constant. A somewhat a less restrictive assumption requires $\nabla \bar{h}$ to vary slowly in space, i.e. to have negligibly small derivatives [2]. Then one can localize expression (3) as

$$\mathbf{r}^{(1)}(\mathbf{x}) \approx K_g \sigma_Y^2 \mathbf{A}^{(1)}(\mathbf{x}) \nabla \bar{h}^{(0)}(\mathbf{x}), \quad \mathbf{A}^{(1)}(\mathbf{x}) = \int_{\Omega} \rho_Y(\mathbf{y}, \mathbf{x}) \nabla_{\mathbf{x}} \nabla_{\mathbf{y}}^T G(\mathbf{y}, \mathbf{x}) d\mathbf{y}. \quad (6)$$

Under these conditions, retaining the two leading terms in the asymptotic expansion of the mean Darcy flux, $\bar{\mathbf{q}} \approx \bar{\mathbf{q}}^{[1]} \equiv \bar{\mathbf{q}}^{(0)} + \mathbf{q}^{(1)}$, yields

$$-\frac{\bar{\mathbf{q}}^{[1]}(\mathbf{x})}{K_g} = \nabla \bar{h}^{(1)}(\mathbf{x}) + \left[\mathbf{I} + \sigma_Y^2 \left(\frac{1}{2} \mathbf{I} - \mathbf{A}^{(1)}(\mathbf{x}) \right) \right] \nabla \bar{h}^{(0)}(\mathbf{x}). \quad (7)$$

For flow through infinite, statistically homogeneous porous media under mean uniform flow conditions, or at points away from boundaries and singularities, the mean hydraulic head gradient $\bar{\mathbf{J}} = \nabla \bar{h}^{(0)} = \text{const}$ and $\nabla \bar{h}^{(i)} = 0$ ($i \geq 1$) [2, 23]. This gives rise to the effective conductivity given approximately by

$$K_{ef}^{[1]} \equiv K_{ef}^{(0)} + K_{ef}^{(1)} = K_g \left[1 + \left(\frac{1}{2} - \frac{1}{d} \right) \sigma_Y^2 \right] \quad (8)$$

where d is the space dimension.

Various attempts to generalize this asymptotic expansion to highly heterogeneous formations were attempted by conjecturing that expression (8) represents the two leading terms in the expansion of an exponent [15, 19],

$$K_{ef} = K_g \exp \left[\left(\frac{1}{2} - \frac{1}{d} \right) \sigma_Y^2 \right]. \quad (9)$$

In recent years the question of validity of expression (9) was the focus of a thorough investigation. It was proven that expression (9) is rigorously valid under one-dimensional flow in log-normal fields where it yields the harmonic mean $K_h = K_g \exp(-\sigma_Y^2/2)$ [3, 18]. It is also rigorously valid under two-dimensional flow in log-normal, statistically isotropic conductivity fields where it yields the geometric mean K_g [15]. For three-dimensional flow in log-normal, statistically isotropic fields, the second-order (in σ_Y^2) term in (8) was found to be in agreement with the Taylor series expansion of (9) [3]. While unsuccessful attempts to prove (9) for three-dimensional flows in such fields have been

reported [14, 17], De Wit [4] demonstrated that the third-order correction in (8) is not equal to the third-order term in the Taylor expansion of (9), thereby proving this conjecture to be not strictly valid for three-dimensional Gaussian isotropic media. Instead, it was demonstrated that this and higher-order terms depend on the shape of the correlation function ρ_Y .

These results suggest that it would be beneficial to view the equation (8) and its higher order terms as a perturbation expansion of the true transport variable in powers of the variance σ_Y^2 . In this sense, the passage from (8) to (9) is a resummation procedure. It thus makes full sense to ask what could be the most general and robust resummation that can generalize (8) in order to extend its domain of validity in the regime of large σ_Y^2 where the initial perturbation expansion breaks down.

It is often the case that perturbation expansions are not converging but are instead diverging series. Even if the series is convergent for small perturbation parameters σ_Y^2 , one is in general interested in the regime where σ_Y^2 is of order 1 and larger. In this case, the perturbation series is divergent and is of no direct use. The study of such summation of divergent series is the problem of great importance in theoretical physics, applied mathematics and engineering. This is because realistic problems are usually solved by means of some calculational algorithm often resulting in divergent sequence of approximations. Assigning a finite value to the limit of a divergent sequence is called renormalization or summation technique. The most widely used such technique is Padé summation [1]. However, the Padé summation method has several shortcomings. First of all, to reach a reasonable accuracy of Padé approximants, one needs to possess tens of terms of a perturbation series. In contrast, only a few terms are often available because of the complexity of the problem. Second, Padé approximants are defined for the series of integer powers. But in many cases asymptotic series arise having noninteger powers. Third, there are quite simple examples that are not Padé summable even for a sufficiently small variable. Last but not least, Padé summation is more of a numerical technique providing answer in the form of numbers. Therefore, it is difficult, if possible, to analyze the results when the considered problem contains several parameters to be varied, since for each given set of parameters one has to repeat the whole procedure of constructing a table of Padé approximants and of selecting from them one corresponding to a visible saturation of numerical values.

We thus turn to the method of so-called self-similar approximation or functional renormalization that provide a very interesting alternative. We first summarize the idea of the technique and then apply it to calculate properties of transport in porous media in the limit of large heterogeneity.

3 General formulation of the self-similar approximation theory

General ideas and the mathematical foundation of the self-similar approximation theory have been described in detail in [7, 24, 25, 26, 27, 9, 10, 11, 8, 28, 29]. The approach is applicable in all cases, when either just a few terms of a series are known or when a number of such terms are available. We are always able to obtain analytical formulas that are easy to consider with respect to varying characteristic parameters. We now expose the general idea of the method of self-similar approximation.

Consider the case, when for a sought function $f(x)$, one derives an approximate perturbative expansion

$$p_k(x) = \sum_{n=0}^k a_n x^{\alpha_n}, \quad (10)$$

in which α_n is an arbitrary real number, integer or noninteger, positive or negative. Following the method of the algebraic self-similar renormalization [7], we define the algebraic transform

$$P_k(x, s) \equiv x^s p_k(x) = \sum_{n=0}^k a_n x^{s+\alpha_n}, \quad (11)$$

where s is real. Rather than constructing a trajectory in the functional space of the initial approximations, the idea behind the introduction of the transform (11) is to deform smoothly the initial functional space of the approximations

$p_k(x)$ in order to obtain a faster and better controlled convergence in the space of the modified functions $P_k(x, s)$. This convergence can then be mapped back to get the relevant estimations and predictions. The exponent s depends on x in general and will be acted upon as a control function in order to accelerate convergence.

Then, by means of the equation $P_0(x, s) = a_0 x^{s+\alpha_0} = \varphi$, we obtain the expansion function $x(\varphi, s) = \left(\frac{\varphi}{a_0}\right)^{1/(s+\alpha_0)}$. Substituting the latter into (10), we have

$$y_k(\varphi, s) \equiv P_k(x(\varphi, s), s) = \sum_{n=0}^k a_n \left(\frac{\varphi}{a_0}\right)^{(s+\alpha_n)/(s+\alpha_0)}. \quad (12)$$

The family $\{y_k\}$ of transforms (10) is called the approximation cascade, since its trajectory $\{y_k(\varphi, s) \mid k = 0, 1, 2, \dots\}$ is bijective to the sequence $\{P_k(x, s) \mid k = 0, 1, 2, \dots\}$ of approximations (11). A cascade is a dynamical system in discrete time $k = 0, 1, 2, \dots$, whose trajectory points satisfy the semigroup property $y_{k+p}(\varphi, s) = y_k(y_p(\varphi, s), s)$. The physical meaning of the above semigroup relation can be understood as the property of functional self-similarity with respect to the varying approximation number. The self-similarity relation is a necessary condition for the fastest convergence criterion.

For the approximation cascade $\{y_k\}$, defined by transform (12), the cascade velocity is

$$v_k(\varphi, s) \equiv y_k(\varphi, s) - y_{k-1}(\varphi, s) = a_k \left(\frac{\varphi}{a_0}\right)^{(s+\alpha_k)/(s+\alpha_0)}. \quad (13)$$

This is to be substituted into the evolution integral

$$\int_{P_{k-1}}^{P_k^*} \frac{d\varphi}{v_k(\varphi, s)} = \tau, \quad (14)$$

in which $P_k = P_k(x, s)$ and τ is the minimal time needed for reaching a fixed point $P_k^* = P_k^*(x, s, \tau)$. Integral (14) with velocity (13) yields

$$P_k^*(x, s, \tau) = \left[P_{k-1}^{-\nu}(x, s) - \frac{\nu a_k \tau}{a_0^{1+\nu}} \right]^{-1/\nu}, \quad (15)$$

where $\nu = \nu_k(s) \equiv \frac{\alpha_k - \alpha_0}{s + \alpha_0}$. Taking the algebraic transform inverse to (11), we find

$$p_k^*(x, s, \tau) \equiv x^{-s} P_k^*(x, s, \tau) = \left[p_{k-1}^{-\nu}(x) - \frac{\nu a_k \tau}{a_0^{1+\nu}} x^{s\nu} \right]^{-1/\nu}. \quad (16)$$

Exponential renormalization [24, 26] corresponds to the limit $s \rightarrow \infty$, at which $\lim_{s \rightarrow \infty} \nu_k(s) = 0$, $\lim_{s \rightarrow \infty} s\nu_k(s) = \alpha_k - \alpha_0$. Then (16) gives

$$\lim_{s \rightarrow \infty} p_k^*(x, s, \tau) = p_{k-1}(x) \exp \left(\frac{a_k}{a_0} \tau x^{\alpha_k - \alpha_0} \right). \quad (17)$$

Accomplishing exponential renormalization of all sums appearing in expression of type (17), we follow the bootstrap procedure [24] according to the scheme $p_k(x) \rightarrow p_k^*(x, s, \tau) \rightarrow F_k(x, \tau_1, \tau_2, \dots, \tau_k)$, with $k \geq 1$.

Let us mention a recent innovation [9] that may improve significantly the convergence of the method, based on the determination of the control parameters from the knowledge of some moments of the function to reconstruct. Let us assume that we can obtain the first $j - 1$ moments $\mu_i, i = 1, 2, \dots, j$ of the sought function $\phi(t)$, in some interval T ,

$$\mu_i = \int_0^T t^{i-1} \phi(t) dt, \quad (18)$$

so that for $j=2$ both zero and first moments are available etc... One can condition the control parameters $\tau_1, \tau_2, \dots, \tau_j$ as follows

$$\int_0^T f_j^*(t, \tau_1, \tau_2, \dots, \tau_{j-1}) t^{i-1} dt = \mu_i, \quad (19)$$

Based on these conditions, one can attempt to solve two different problems, the first one corresponds to an approximate reconstruction of the function $\phi(t)$ within the same interval $[0, T]$ where moments are given or measured. The second problem consists in extrapolating to $t > T$. It is also possible to use an hybrid approach, where some controls are obtained from the agreement with expansion, while the remaining ones are found from the conditions on moments.

4 Resummation of lower order expansions in σ_Y^2 .

4.1 What can be extracted from the expansion of $K_{ef}^{[1]}$

Assume that the following extremely short expansion has been obtained,

$$K(\sigma_Y) \simeq 1 - a\sigma_Y^2, \quad a = \frac{1}{d} - \frac{1}{2} \quad (\sigma_Y^2 \rightarrow 0). \quad (20)$$

Consider the case $a > 0$. In order to find the behavior of $K(\sigma_Y)$ for arbitrary σ_Y^2 , we continue it from the region of $\sigma_Y^2 \rightarrow 0$ self-similarly, along the most stable trajectory, with the crossover index s , determined by the condition of the minimum of the multiplier [7, 24, 29]

$$m(\sigma_Y, s) = 1 - a\sigma_Y^2 \frac{1+s}{s}, \quad (21)$$

from where

$$s(\sigma_Y) = a\sigma_Y^2 (1 - a\sigma_Y^2)^{-1}, \quad \sigma_Y < a^{-1/2}, \\ s \rightarrow \infty, \quad \sigma_Y \geq a^{-1/2},$$

corresponding to the self-similar approximation

$$K^*(\sigma_Y) = \left(\frac{s(\sigma_Y)}{s(\sigma_Y) + a\sigma_Y^2} \right)^{s(\sigma_Y)} = (2 - a\sigma_Y^2)^{\frac{a\sigma_Y^2}{a\sigma_Y^2 - 1}}, \quad \sigma_Y < a^{-1/2}, \quad (22)$$

$$K^*(\sigma_Y) = \exp(-a\sigma_Y^2), \quad \sigma_Y \geq a^{-1/2}. \quad (23)$$

This suggests one self-similar expression (22) up to $\sigma_{Y0} = a^{-1/2}$, and another (23), exponentially “soft”, above this value. Strictly speaking, formula (22) is applicable for σ_Y only up to $(2/a)^{1/2}$, where it predicts a spurious zero of conductivity. In this particular case, the self-similar approximation plausibly reconstructs the exponential function for arbitrary σ_Y , even in the absence of any a priori assumption on the asymptotic behavior at $\sigma_Y \rightarrow \infty$.

For negative a ($d > 2$), there is no limiting value σ_{Y0} and this yields

$$K^*(\sigma_Y) = \left(\frac{s(\sigma_Y)}{s(\sigma_Y) + a\sigma_Y^2} \right)^{s(\sigma_Y)}, \quad (24)$$

which can be used for arbitrary σ_Y . This expression appears to be more stable than the previously proposed exponential solution $\exp(-a\sigma_Y^2)$ for arbitrary σ_Y , as indicated by the analysis of the multipliers. Expression (24) also predict a smaller conductivity than the exponential function for arbitrary σ_Y suggesting that, for the most interesting case $d = 3$, the ansatz equation (8) should be replaced. Analysis of higher-order expansions will provide more details on the sought function.

4.2 Resummation of the second-order expansion in σ_Y^2 . One-parameter formula

Available from [3] in the next order in σ_Y^2 , we have

$$K_{ef}^{[2]} \equiv K_{ef}^{[1]} + K_{ef}^{(2)} = K_g \left[1 + \left(\frac{1}{2} - \frac{1}{d} \right) \sigma_Y^2 + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{d} \right)^2 \sigma_Y^4 \right]. \quad (25)$$

Below, for simplicity, we apply our resummation technique to the dimensionless quantity

$$K(z) \simeq 1 + a_1 z + a_2 z^2, \quad z \equiv \sigma_Y^2, \quad a_1 = \left(\frac{1}{2} - \frac{1}{d} \right), \quad a_2 = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{d} \right)^2. \quad (26)$$

Application of accuracy-through-order conditions, or of the superexponential approximants give, almost trivially, an exponential solution. Note that the Padé approximant available in this case,

$$P(z) = \frac{1 + (-a_2/a_1 + a_1)z}{1 - a_2/a_1 z}, \quad (27)$$

for $d = 3$, possesses a singularity at $z = 12$, which is wrong.

The set of approximations to $K(z)$, including the two starting terms from (26), can be written down as follows:

$$K_0 = 1,$$

$$K_1 = 1 + a_1 z,$$

and the expression for the renormalized quantity a_1^* can be readily obtained:

$$K_1^* = \left(\frac{s_1}{s_1 - a_1 z} \right)^{s_1} \implies \left(\frac{s_1}{-a_1} \right)^{s_1} z^{-s_1} (z \rightarrow \infty), \quad (28)$$

where the stabilizer s_1 should be negative, if we want to reproduce in the limit of $z \rightarrow \infty$, the correct, supposedly power-law behavior of the conductivity. A different set of approximations, which does not include the constant term from (26) into the renormalization procedure, has the form:

$$\overline{K}_1 = a_1 z, \quad (29)$$

$$\overline{K}_2 = a_1 z + a_2 z^2, \quad (30)$$

and applying the standard procedure of [24, 25], we obtain

$$K_2^* = 1 + a_1 z \left[1 - \frac{a_2 z}{a_1(1 + s_2)} \right]^{-(1+s_2)} \implies \left(-\frac{a_2}{1 + s_2} \right)^{-(1+s_2)} a_1^{2+s_2} z^{-s_2} (z \rightarrow \infty). \quad (31)$$

Demanding now that both (28) and (31) have the same power-law behavior at $z \rightarrow \infty$, we find that

$$s_2 = s_1 \equiv s$$

Requiring now the fulfillment of the stability criteria for the two available approximations in the form of the minimal-difference condition (see section 3), we obtain the condition that the *negative* stabilizer s should be determined from the *minimum* of the expression:

$$\left| \left[\left(\frac{-a_2}{1 + s} \right)^{-(1+s)} a_1^{(2+s)} - \left(\frac{s}{-a_1} \right)^s \right] \right|. \quad (32)$$

Generally speaking, it is sufficient to ask for an extremum of this difference.

In the case of $d = 3$, the maximum is located at the point $s = -1.218$. The final formulae have the following form:

$$K_1^*(z) = \left(\frac{s}{s - a_1 z} \right)^s, \quad (33)$$

$$K_2^*(z) = 1 + a_1 z \left[1 - \frac{a_2 z}{a_1(1+s)} \right]^{-(1+s)}. \quad (34)$$

This last formula gives a lower bound for conductivity while the upper bound is simply $\exp(a_1 z)$ (equation (8)) which, in this case, is the only available “factor”-approximant based on all available (three) terms from the expansion. The corresponding multiplier is

$$M_2^*(z) = \frac{a_1(1+s) + a_2 z s}{a_1(1+s) - a_2 z} \left(1 - \frac{a_2}{a_1(1+s)} z \right)^{-(s+1)} \quad (35)$$

and the weighted average [27, 9] is given by

$$C(z) = \frac{1 + K_2^*(z) |M_2^*(z)|^{-1}}{\exp(-a_1 z) + |M_2^*(z)|^{-1}}, \quad (36)$$

providing the one-parametric formula for 3d-conductivity. See Fig.1 for a comparison of different formulas for the conductivity as a function of the variance $z \equiv \sigma_Y^2$ defined in equation (26). The solid line corresponds to the average $C(z)$, while the dotted line presents $K_2^*(z)$. The dashed line is the celebrated exponential Landau-Lifshitz-Matheron (LLM) conjecture. The dash-dotted line corresponds to the result of resummation based on first-order expansion, $K^*(\sigma_Y)$.

4.3 Resummation of the third-order expansion in σ_Y^2 . Two-parameters formula

The expansion to the next order is given by [4],

$$K(z) \simeq 1 + a_1 z + a_2 z^2 + a_3 z^3, \quad a_3(Z) = \frac{1}{6} \left(\frac{1}{2} - \frac{1}{d} \right)^3 - Z, \quad (37)$$

where $Z = 0.0042/3$ (in the case of a Gaussian covariance), or $Z = 0.0014/3$ (in the case of an exponentially decaying covariance).

Using all terms from the expansion, we can create the following “odd” factor-approximant [11],

$$K_3^*(z, Z) = 1 + a_1 z \left(1 - \frac{a_2}{a_1(s_2(Z) + 1)} z \right)^{-(s_2(Z)+1)}, \quad s_2(Z) = -2 \frac{a_2^2 - a_3(Z)a_1}{a_2^2 - 2a_3(Z)a_1}, \quad (38)$$

while $K_2^*(z) = \exp(a_1 z)$, which recovers expression (8). The approximant $K_2^*(z)$ gives an upper bound for the conductivity coefficient, while $K_3^*(z, Z)$ given by (38) provides a lower bound. The corresponding multipliers can be readily written down,

$$M_3^*(z, Z) = \frac{a_1(1 + s_2(Z)) + a_2 z s_2(Z)}{a_1(1 + s_2(Z)) - a_2 z} \left(1 - \frac{a_2}{a_1(s_2(Z) + 1)} z \right)^{-(s_2(Z)+1)}, \quad (39)$$

$$M_2^*(z) = \exp(a_1 z). \quad (40)$$

This allows us to obtain the weighted average of the conductivity coefficient

$$C(z, Z) = \frac{1 + K_3^*(z, Z) |M_3^*(z, Z)|^{-1}}{\exp(-a_1 z) + |M_3^*(z, Z)|^{-1}}, \quad (41)$$

providing a two-parameters formula for the 3d-conductivity. The results for $C(z, Z)$ for Gaussian and exponential covariances are shown in Fig. 2, with the dotted line for the exponential case and with the solid line for the Gaussian case. These results are compared with the Landau-Lifshitz-Matheron (LLM) shown with dashed line.

Numerical data in the exponential case are available till $z = 7$ [16] and in the Gaussian case up to $z = 6$ [5]. Our results suggest that all formulas based on the expansion on σ_Y^2 – up to the third order underestimate the conductivity and more terms are needed to improve the accuracy of the resummed expressions.

A different approach aimed at increasing the accuracy consists in getting expressions for the conductivity in the limit of $\sigma_Y^2 \rightarrow \infty$, for instance from expansions in inverse powers of σ_Y^2 . It is known (see Ref.[8, 28, 29]) that when the asymptotic form of the solution is known, even only qualitatively, the formulas for the sought function can be improved very significantly. Even the knowledge of the leading power in the limit of large σ_Y^2 would be of utmost importance.

5 $1/d$ -expansion and resummation

5.1 One-parametric case (d=3)

Expression (26) for $K(z)$ in the second order of perturbation theory can be re-written in the form of an expansion in the parameter $1/d$ with coefficients dependent on z ,

$$K(m) \simeq b_0(z) + b_1(z)m + b_2(z)m^2, \quad m \equiv 1/d; \quad (42)$$

$$b_0(z) = 1 + \frac{z}{2} + \frac{z^2}{8}, \quad b_1(z) = -z - \frac{z^2}{2}, \quad b_2(z) = \frac{z^2}{2}. \quad (43)$$

The theory of self-similar super-exponential approximants [24, 26, 9, 10] then provides the following approximant

$$K_2^*(m) = b_0 \exp \left(\frac{b_1}{b_0} \tau_1 m \exp \left(\frac{b_2}{b_1} \tau_2 m \right) \right), \quad \tau_1 = 1, \quad \tau_2 = 1 - \frac{b_1^2}{2b_0b_2}. \quad (44)$$

$K_2^*(m)$ is located within the bounds given by $K_2^*(z)$ and $\exp(a_1 z)$ and provides a one-parametric formula for the $d = 3$ -conductivity, as shown in Fig. 3. $K_2^*(m)$ (dotted line) appears to be located within the bounds outlined by $C(z)$ (dashed) and $\exp(a_1 z)$ (solid line) and provides a one-parametric formula for the conductivity in three dimensions. The perturbative expression $K(m)$ (dashed-dot line) is shown as well for comparison.

5.2 Two-parametric case (d=3)

Expression for $K(z)$ (37) in the third order of perturbation theory can also be re-written in the form of an $1/d$ -expansion with coefficients dependent on z and Z ,

$$K(m, Z) \simeq b_0(z, Z) + b_1(z)m + b_2(z)m^2 + b_3(z)m^3; \quad (45)$$

$$b_0(z, Z) = 1 + \frac{z}{2} + \frac{z^2}{8} + \left(\frac{1}{48} - Z \right) z^3, \quad b_1(z) = -z - \frac{z^2}{2} - \frac{z^3}{8}, \quad b_2(z) = \frac{z^2}{2} + \frac{z^3}{4}, \quad (46)$$

$$b_2(z) = \frac{z^2}{2} + \frac{z^3}{4}, \quad b_3(z) = -\frac{z^3}{6}. \quad (47)$$

We apply the technique of self-similar superexponential function in its variant detailed in [9, 10], giving the following approximants

$$K_2^*(m, Z, \tau_1, \tau_2) = b_0 \exp \left(\frac{b_1}{b_0} \tau_1 m \exp \left(\frac{b_2}{b_1} \tau_2 m \right) \right), \quad (48)$$

$$K_3^*(m, Z, \tau_1, \tau_2, \tau_3) = b_0 \exp \left(\frac{b_1}{b_0} \tau_1 m \exp \left(\frac{b_2}{b_1} \tau_2 m \exp \left(\frac{b_3}{b_2} \tau_3 m \right) \right) \right), \quad (49)$$

$$\tau_3 = \frac{1}{6} \frac{(-b_1^4 + 6b_0^2 b_1 b_3 - 6\tau_2 b_0 b_1^2 b_2 - 3\tau_2^2 b_0^2 b_2^2)}{\tau_2 b_0^2 b_1 b_3}. \quad (50)$$

In order to check whether the sequence of K_j^* converges, we study their mapping multipliers, $M_j^*(t, \tau_1, \tau_2, \dots, \tau_j)$ defined as

$$M_j^*(m, Z, \tau_1, \dots, \tau_j) \equiv \frac{1}{b_1} \frac{\partial}{\partial t} K_j^*(m, Z, \tau_1, \dots, \tau_j). \quad (51)$$

This definition of the multipliers allows us to compare the convergence of the expansion and of the renormalized expressions, making clear what can be expected a priori.

This provides a matrix of self-similar approximants, indexed by the order j and by the number of control parameters,

$$K_{21}^*(m, Z) = K_j^*(m, Z, \tau_1, 1), \quad K_{22}^*(m, Z) = K_j^*(m, Z, \tau_1, \tau_2), \quad (52)$$

$$K_{31}^*(m, Z) = K_3^*(m, Z, \tau_1, 1, 1), \quad K_{32}^*(m, Z) = K_3^*(m, Z, \tau_1, \tau_2, 1), \quad K_{33}^*(m, Z) = K_3^*(m, Z, \tau_1, \tau_2, \tau_3), \quad (53)$$

Approximants $K_{22}^*(m, Z)$ and $K_{32}^*(m, Z)$ form the closest pair. Their average

$$K^*(m, Z) = \frac{K_{22}^*(m, Z) |M_{22}^*(m, Z)|^{-1} + K_{32}^*(m, Z) |M_{32}^*(m, Z)|^{-1}}{|M_{22}^*(m, Z)|^{-1} + |M_{32}^*(m, Z)|^{-1}}. \quad (54)$$

is located within the bounds given by $K_3^*(z, Z)$ (38) and $\exp(a_1 z)$ and provides a useful formula for the conductivity coefficient. The results for the exponential and Gaussian covariances are shown in Figures 4 and 5 respectively. In the exponential case, good agreement between $K^*(m, Z)$ (dash-dot) and the Landau-Lifshitz-Matheron (LLM) conjecture (solid) remains valid till $z \approx 11$. The average behavior appears to be located within the bounds outlined by $K_{22}^*(m, Z)$ (dashed) and $K_{32}^*(m, Z)$ (dotted) and provides a reasonable formula for the conductivity. Numerical data are available till $z = 7$ [16] and they agree well with our lower bound.

In the Gaussian case $K_{32}^*(m, Z)$ (dotted line) gives an approximation which is closer to the Landau-Lifshitz-Matheron (LLM) conjecture (solid) than $K_{22}^*(m, Z)$ (dashed line). In this case, numerical results are available up to $z = 6$ [5].

We conclude, tentatively, that all formulas based on $1/d$ -expansion in the third order provide rather accurate expressions for conductivity for small and moderate variances and disagree with the LLM-conjecture for very large variances. To the best of our knowledge, the region of large variances is not accessible by other techniques, numerically or theoretically. All formulas based on the novel proposed $1/d$ -expansion in the third order provide rather accurate expressions for the conductivity coefficient. We note that $1/d$ -expansions may be faster converging than the original expansion in variances and should be investigated future.

6 Future directions

We have shown that it is possible to extend the moment equation approach using the self-similar functional renormalization method in order to provide a stable and robust estimation of the transport variables of the three-dimensional medium at large values of the perturbation parameter σ_Y^2 .

Future directions include the extension of the self-similar functional renormalization method to go beyond the characterization of the heterogeneity solely in terms of the variance and consider also the dependence of the transport properties with respect to the skewness (third normalized cumulant) and kurtosis (fourth normalized cumulant).

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References

- [1] Baker, G.A., Jr. and P. Graves-Morris, Padé approximants, 2nd edition (Cambridge University Press, Cambridge, 1996)
- [2] Dagan G. Flow and transport in porous formations. New York: Springer-Verlag, 1989.
- [3] Dagan G. Higher-order correction of effective conductivity in heterogeneous formations of lognormal conductivity distribution. *Transport in Porous Media* 12: 279-290, 1993
- [4] De Wit A. Correlation structure dependence of the effective permeability of heterogeneous porous media. *Phys Fluids* 7(11):2553-2562, 1995
- [5] Dykaar B., and Kitanidis P., Determination of the Effective Hydraulic Conductivity for Heterogeneous Porous Media Using a Numerical Spectral Approach 2. Results, *Water Resour Res* 28 (4): 1167-1178, 1992
- [6] Feng, S., Halperin, B.I. and Sen, P.N., Transport properties of continuum systems near the percolation threshold, *Phys. Rev. B* **35**, 197–214, 1987
- [7] Gluzman S., and Yukalov V.I., Algebraic Self-Similar Renormalization in Theory of Critical Phenomena, *Phys. Rev. E* 55, 3983-3999, 1997
- [8] Gluzman S., and Yukalov V. I., Unified Approach to Crossover Phenomena, *Phys. Rev. E* **58**, 4197-4209, 1998
- [9] Gluzman S., and Sornette D., Classification of Possible Finite-Time Singularities by Functional Renormalization, *Phys. Rev. E* 6601 N1 PT2:U315-U328, 2002
- [10] Gluzman S, Sornette D., and Yukalov V. I., Reconstructing Generalized Exponential Laws by Self-Similar Exponential Approximants, *Int. J. of Mod. Phys.C*, to appear, cond-mat/0204326 , 2002
- [11] Gluzman S., Yukalov V. I., and Sornette D., Self-Similar Factor Approximants, *Phys. Rev. E*, to appear, cond-mat/0208486, 2003
- [12] Guadagnini A, Neuman SP. Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains, 2, computational examples. *Water Resour Res* 35(10): 3019-3039, 1999
- [13] Hsu K-C, Zhang D, Neuman SP. Higher-order effects on flow and transport in randomly heterogeneous porous media. *Water Resour Res* 32(3): 571-582, 1995
- [14] King PR. The use of renormalization for calculating effective hydraulic conductivity. *Transport in Porous Media* 4: 37-58, 1989
- [15] Matheron G., Elements pour une theorie des milieux poreux. Masson et Cie, Paris, 1967.
- [16] Neuman, S. and S. Orr, Prediction of Steady State Flow in Nonuniform Geologic Media by Conditional Moments: Exact Nonlocal Formalism, Effective Conductivities, and Weak Approximation, *Water Resour Res* 29 (2): 341-364, 1993
- [17] Noetinger B. The effective permeability of a heterogeneous porous medium. Institute Francais du Petrole, Division Gisements. Project: B-4332044, 1990.
- [18] Paleologos EK, Neuman SP, Tartakovsky DM. Effective hydraulic conductivity of bounded, strongly heterogeneous porous media. *Water Resour Res* 32(5): 1333-1341, 1996
- [19] Shvidler MI. Flow in heterogeneous media. *Izv. Akad. Nauk SSSR. Mech. Zhidk. Gaza* 3: 185, 1962

- [20] Sornette, D., Critical transport and failure exponents in continuum crack percolation, *J.Physique (Paris)* **49**, 1365-1377 1988.
- [21] Sornette, D., *Critical Phenomena in Natural Sciences* (Springer Series in Synergetics, Heidelberg, 2000).
- [22] Tartakovsky, D.M. and Neuman, S.P., Transient flow in bounded randomly heterogeneous domains: 1. Exact conditional moment equations and recursive approximations. *Water Resour Res* **34**(1): 1-12, 1998
- [23] Tartakovsky, D.M. and Neuman, S.P., Transient effective hydraulic conductivity under slowly and rapidly varying mean gradients in bounded three-dimensional random media. *Water Resour Res* **34**(1): 21-32 1998.
- [24] Yukalov V.I., and Gluzman S., Self-Similar Bootstrap of Divergent Series, *Phys. Rev. E* **55**, 6552-6570, 1997
- [25] Yukalov V. I., and Gluzman S., Critical Indices as Limits of Control Functions, *Phys. Rev. Lett.* **79** , 333-336, 1997
- [26] Yukalov V. I., and Gluzman S., Self-Similar Exponential Approximants, *Phys. Rev. E* **58**, 1359-1382, 1998
- [27] Yukalov V. I , and Gluzman S., Weighted Fixed Points in Self-Similar Analysis of Time Series, *Int. J. of Mod. Phys. B* **13**, 1463-1476, 1999
- [28] Yukalov V.I , Yukalova E.P., and Gluzman S., Self-Similar Interpolation in Quantum Mechanics, *Phys. Rev. A* **58**, 96-115, 1998
- [29] Yukalov V. I, and Gluzman S., Self-Similar Crossover in Statistical Physics, *Physica A* **273**, 401-415, 1999

Figures Captions

Figure 1. Dependence of different estimators of the conductivity as a function of the variance $z \equiv \sigma_Y^2$ defined in equation (26). The solid line corresponds to $C(z)$, the dotted line shows $K_2^*(z)$. The dashed line is the Landau-Lifshitz-Matheron (LLM) conjecture. $K^*(\sigma_Y)$ (dash-dotted line) corresponds to the result of resummation based on the first-order expansion.

Figure 2. Dependence of the conductivity $C(z, Z)$ as a function of the variance $z \equiv \sigma_Y^2$ defined in equation (26) for Gaussian and exponential covariances, shown with solid and dotted lines respectively. For comparison, the Landau-Lifshitz-Matheron (LLM) conjecture is shown in dashed lines.

Figure 3. As a function of the variance $z \equiv \sigma_Y^2$ defined in equation (26), this figure shows a comparison between the conductivity $K_2^*(m)$ (dotted line) with $C(z)$ (dashed) and with $\exp(a_1 z)$ (solid line). The Perturbative expansion $K(m)$ is shown with the dashed-dot line.

Figure 4. Exponential covariance: $K^*(m, Z)$ (dash-dot) is compared with the Landau-Lifshitz-Matheron (LLM) conjecture (solid line). Two approximants $K_{22}^*(m, Z)$ (dashed) and $K_{32}^*(m, Z)$ (dotted) are shown as well.

Figure 5. Gaussian case: Approximant $K_{32}^*(m, Z)$ (dotted) compared with the Landau-Lifshitz-Matheron (LLM) conjecture (solid line). The approximant $K_{22}^*(m, Z)$ (dashed line) is shown as well.

Fig. 1

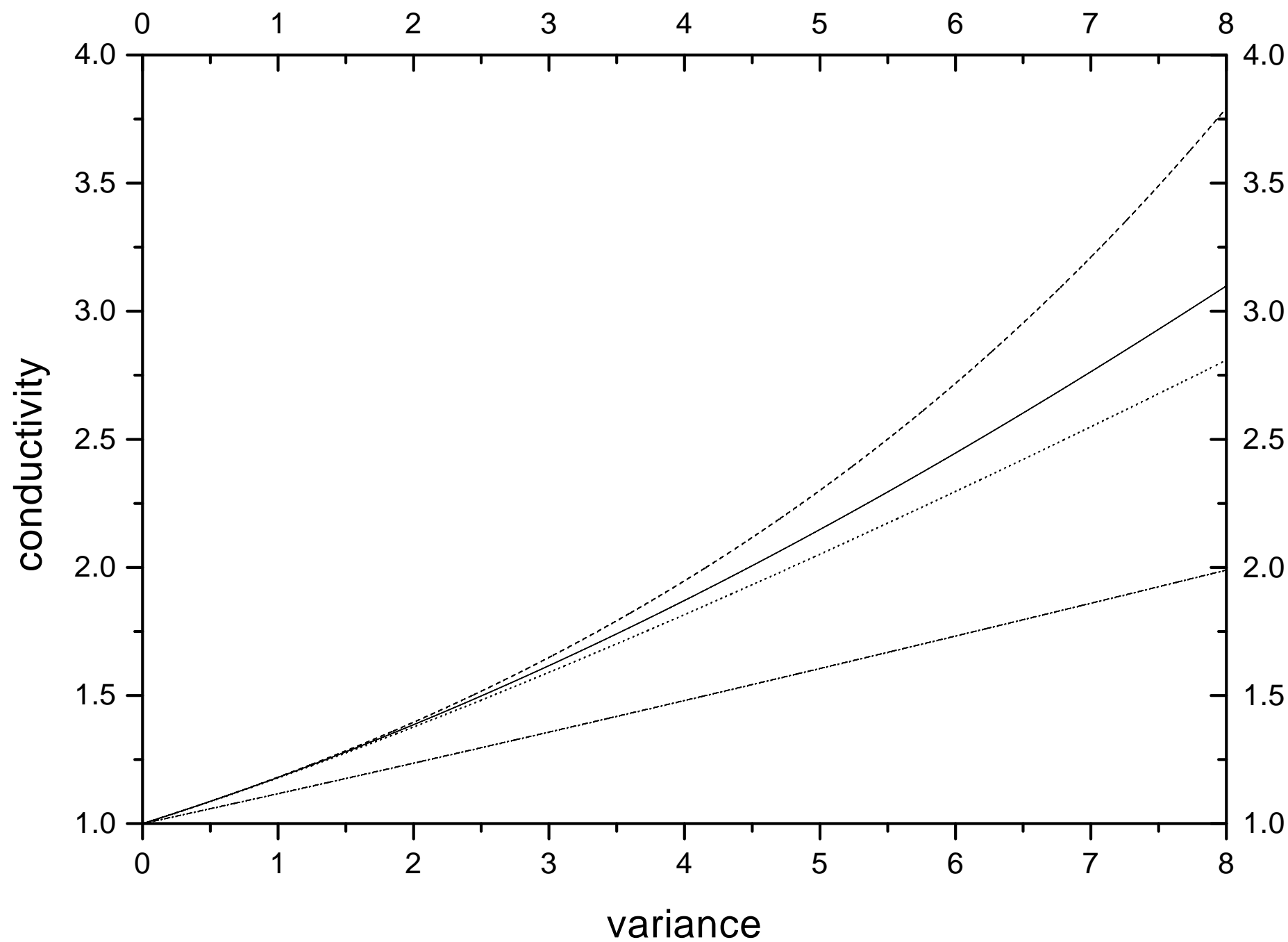


Fig. 2

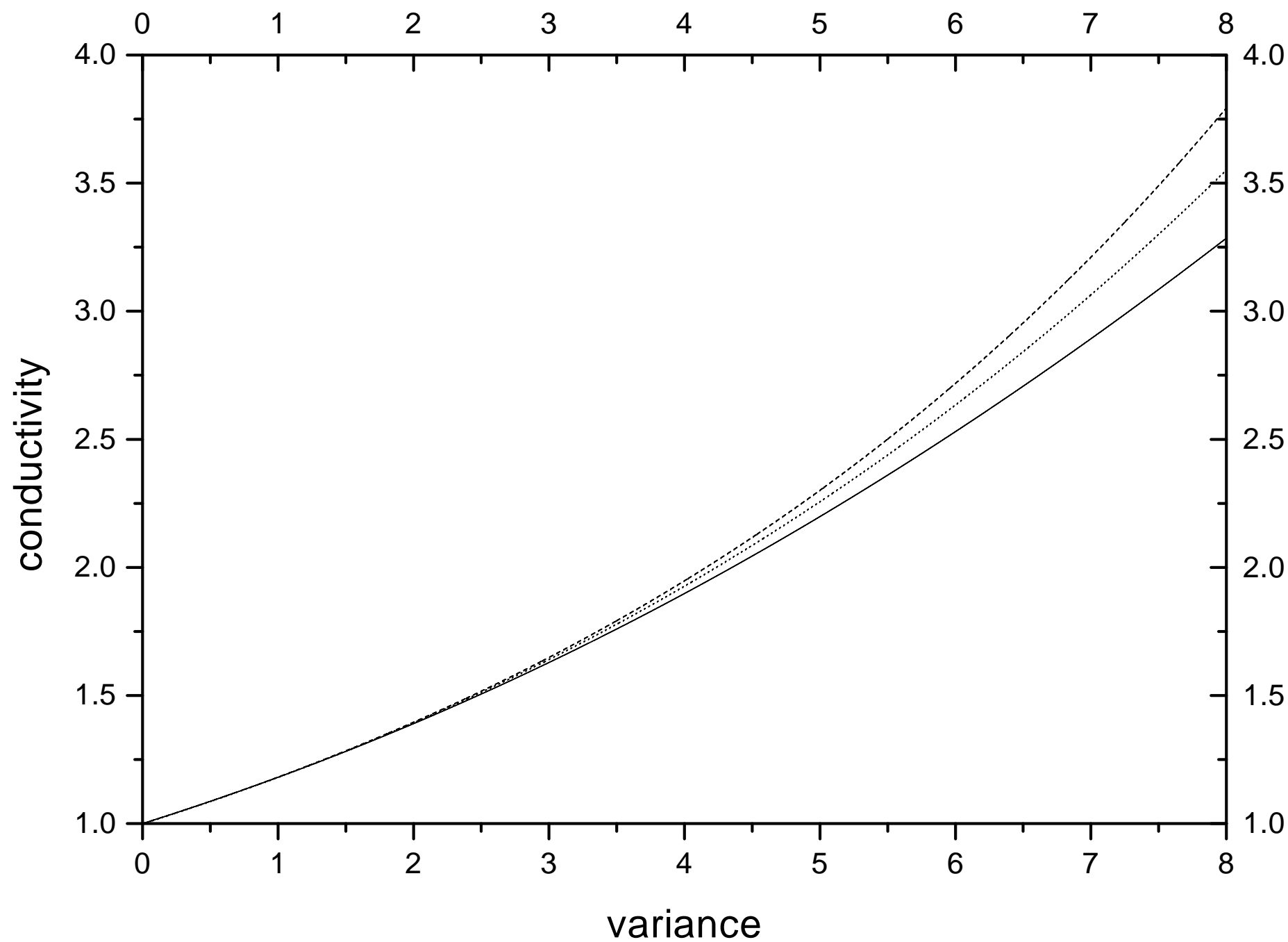


Fig. 3

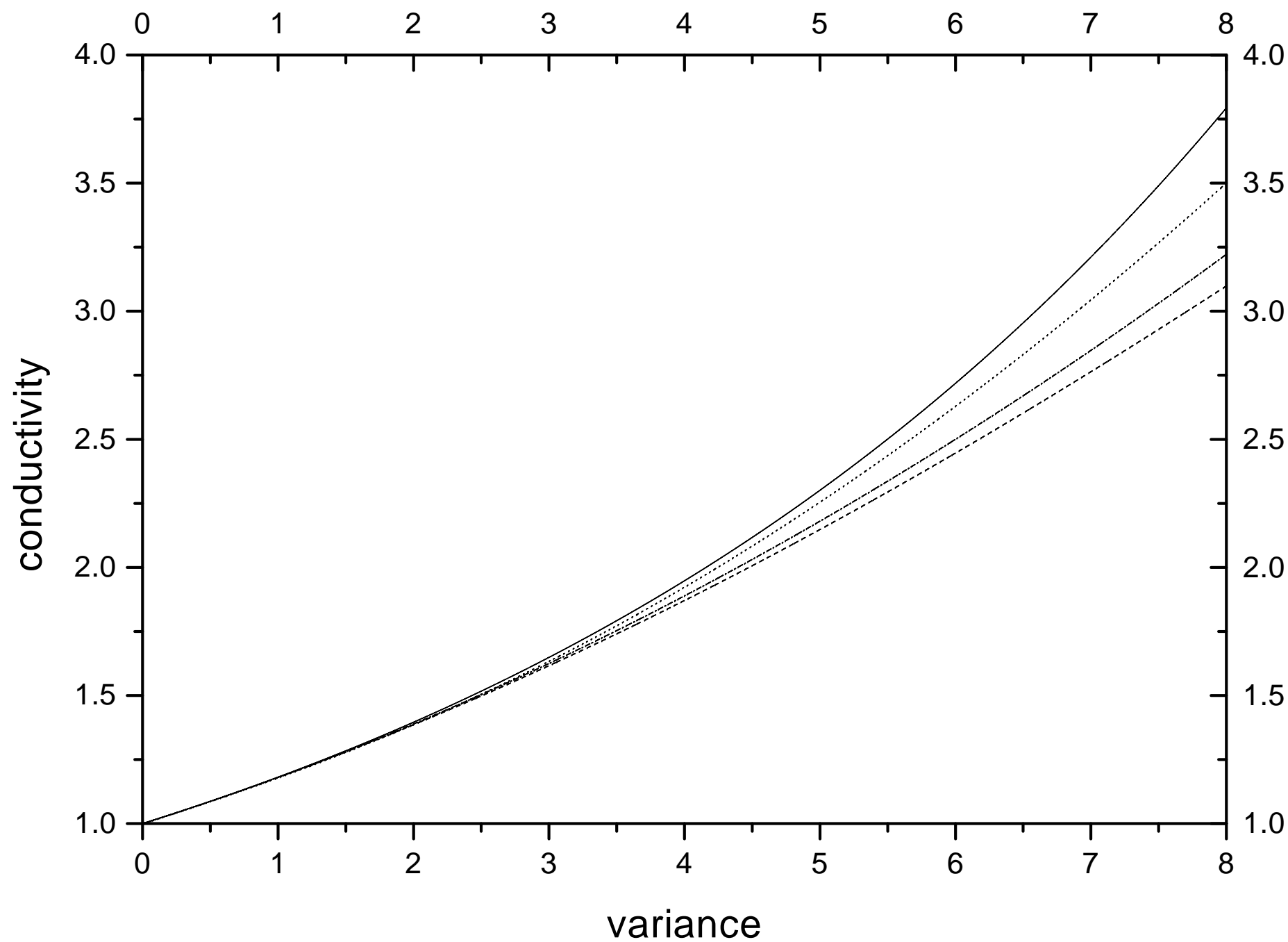


Fig. 4

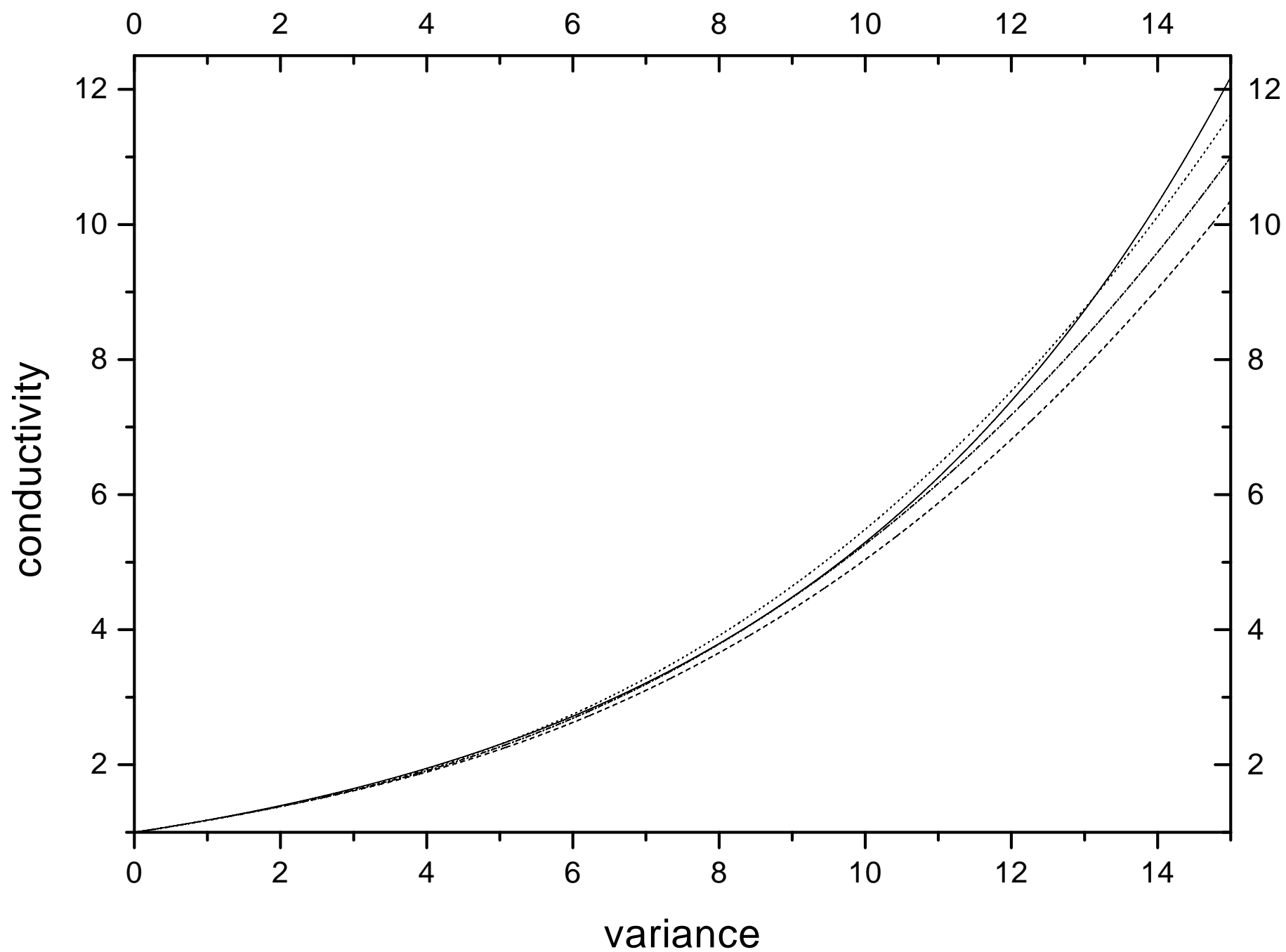


Fig. 5

